On $osp(M|2n)$ integrable open spin chains

D. Arnaudon, N. Crampé, A. Doikou, L. Frappat, É. Ragoucy

Laboratoire d'Annecy-le-Vieux de Physique Théorique LAPTH CNRS, UMR 5108, associée à l'Université de Savoie LAPP, BP 110, F-74941 Annecy-le-Vieux Cedex, France

J. Avan

Laboratoire de Physique Théorique et Modélisation Université de Cergy, 5 mail Gay-Lussac, Neuville-sur-Oise F-95031 Cergy-Pontoise Cedex

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Abstract

We consider open spin chains based on $osp(m|2n)$ Yangians. We solve the reflection equations for some classes of reflection matrices, including the diagonal ones. Having then integrable open spin chains, we write the analytical Bethe Ansatz equations. More details and references can be found in [1, 2].

1 RTT presentation for $osp(M|2n)$ Yangians

Let us consider an $M+2n$ dimensional $\mathbb{Z}_2$-graded vector space, with the M first indices bosonic and the $2n$ last ones fermionic.

Define

$$R(u) = \mathbb{I} + \frac{P}{u} - \frac{Q}{u+\kappa} \,.$$

P being the super permutation, and $Q = P^{t_1} = P^{t_2}$ being P partially transposed.

The R-matrix $R(u)$ satisfies the super Yang–Baxter equation

$$R_{12}(u)\, R_{13}(u+v)\, R_{23}(v) = R_{23}(v)\, R_{13}(u+v)\, R_{12}(u)$$

if $2\kappa = (M-2n-2)$, with a *graded tensor product* .

One defines the Yangian of $osp(M|2n)$ by the generators

$$T(u) = \sum_{n\in\mathbb{Z}_{\geq 0}} T_{(n)}\, u^{-n} \qquad T(0) = \mathbb{I}$$

and the relations

$$R_{12}(u-v)\,T_1(u)\,T_2(v)=T_2(v)\,T_1(u)\,R_{12}(u-v)$$
$$T^t(u-\kappa)\,T(u)=\mathbb{I}$$

(i.e. RTT=TTR relations and "orthogonality relation") [3].

2 Closed chain integrability

The closed chain monodromy matrix is defined by

$$T_a(u)=R_{aL}(u)R_{a,L-1}(u)\cdots R_{a2}(u)R_{a1}(u)$$

Using the Yang–Baxter equation, one proves that the closed chain transfer matrices, given by the super trace $t(u)=\mathrm{Tr}_a\,T_a(u)$, commute for all values of the spectral parameter u:

$$\Big[t(u),t(v)\Big]=0,\qquad\qquad \forall u,v$$

3 Reflection equation and open chain integrability

We consider $K^-(u)\in End(\mathbb{C}^{M+2n})$, solution of the reflection equation:

$$R_{ab}(u_a-u_b)\ K_a^-(u_a)\ R_{ba}(u_a+u_b)\ K_b^-(u_b)\ =\ K_b^-(u_b)\ R_{ab}(u_a+u_b)\ K_a^-(u_a)\ R_{ba}(u_a-u_b)$$

Let

$$T_a(u)=R_{aL}(u)R_{a,L-1}(u)\cdots R_{a2}(u)R_{a1}(u)$$

and

$$\hat{T}_a(u)=R_{1a}(u)R_{2a}(u)\cdots R_{L-1,a}(u)R_{La}(u)$$

The open chain monodromy matrix is the super trace

$$t(u)=\mathrm{Tr}_a\,K_a^+(u)\ T_a(u)\ K_a^-(u)\ \hat{T}_a(u)\,,$$

where ${K^+}^t(-\lambda-i\kappa)$ is another solution of the reflection equation. Again, as was first proved by Cherednik and Sklyanin using the Yang–Baxter and reflection equations, [4, 5], $[t(u),t(v)]=0,\quad \forall u,v$.

4 Solutions of the reflection equation

4.1 Diagonal solutions

We solve the reflection equation for K of the form

$$K(u) = \mathrm{diag}\Big(k_1(u), \cdots, k_M(u)\,;\, k_{M+1}(u), \cdots, k_{M+n}(u)\Big)$$

There are three families of generic diagonal solutions and two particular cases

D1: Solutions of $sl(M+2n)$ type, with one free parameter, for M even

$$\begin{aligned} &k_i(u) = 1\ , \\ &k_{\bar{\imath}}(u) = \frac{1+cu}{1-cu}\ , \qquad \forall\, i \in \{1, ..., \frac{M}{2}; M+1, ..., M+n\} \end{aligned}$$

This solution has no extension to odd M.

D2: Solutions with three different values of $k_l(u)$, depending on one free parameter

$$k_1(u) = \frac{1+c_1 u}{1-c_1 u}\ , \qquad k_M(u) = \frac{1+c_M u}{1-c_M u}\ , \qquad k_j(u) = 1 \quad \forall\, j \neq 1, M$$

where $(\kappa-1)c_1 c_M + c_1 + c_M = 0$. This solution does not hold for $M = 0, 1$.

D3: Solutions without any free continuous parameter, but with two integer parameters m_1, n_1, and $c = \frac{2}{\kappa-(2m_1-2n_1-1)}$

$$\begin{aligned} &k_i(u) = k_{\bar{\imath}}(u) = 1 \qquad \forall\, i \in \{1, ..., m_1; M+1, ..., M+n_1\} \\ &k_i(u) = k_{\bar{\imath}}(u) = \frac{1+cu}{1-cu} \qquad\qquad \text{otherwise} \end{aligned}$$

D4: In the particular case of $so(4)$, the solution takes the more general form:

$$K(u) = \mathrm{diag}\left(1\ ,\ \frac{1+c_2 u}{1-c_2 u}\ ,\ \frac{1+c_3 u}{1-c_3 u}\ ,\ \frac{1+c_2 u}{1-c_2 u}\ \frac{1+c_3 u}{1-c_3 u}\right)$$

This solution contains the three generic solutions D1 ($c_2 c_3 = 0$), D2 ($c_2 + c_3 = 0$) and D3 ($c_2 = c_3 = \infty$).

D5: In the particular case of $so(2)$, any function-valued diagonal matrix is solution.

4.2 Antidiagonal and mixed solutions

The classification of such solutions is best shown by a few examples.
One finds the two following solutions for $osp(4|2)$:
so **diagonal :**

$$
\left(\begin{array}{cccc|cc}
1 & & & & & \\
 & 1 & & & & \\
 & & -1 & & & \\
 & & & -1 & & \\
\hline
 & & & & k_5 & \ell_5 \\
 & & & & \ell_6 & -k_5
\end{array}\right)
$$

where $k_5^2 + \ell_5\ell_6 = 1$.
sp **diagonal :**

$$
\left(\begin{array}{cccc|cc}
1 & & & 0 & & \\
 & 0 & \ell_2 & & & \\
 & \ell_2^{-1} & 0 & & & \\
0 & & & 1 & & \\
\hline
 & & & & 1 & \\
 & & & & & 1
\end{array}\right)
$$

For $osp(2|4)$ the two solutions take the form
so **diagonal :**

$$
\left(\begin{array}{cc|cccc}
1 & & & & & \\
 & -1 & & & & \\
\hline
 & & k_3 & & & \ell_3 \\
 & & & k_4 & \ell_4 & \\
 & & & \ell_5 & -k_4 & \\
 & & \ell_6 & & & -k_3
\end{array}\right)
$$

where $k_3^2 + \ell_3\ell_6 = 1$ and $k_4^2 + \ell_4\ell_5 = 1$.
sp **diagonal :**

$$
\left(\begin{array}{cc|cccc}
0 & \ell_1 & & & & \\
\ell_1^{-1} & 0 & & & & \\
\hline
 & & 1 & & & \\
 & & & -1 & & \\
 & & & & -1 & \\
 & & & & & 1
\end{array}\right)
$$

5 Pseudovacuum and one eigenvalue of the transfer matrix for the open chain

We now choose an appropriate pseudo-vacuum, which is an exact eigenstate of the transfer matrix $t(u)$ of the open chain; it is the state with all "spins" up, i.e.

$$|\omega_+\rangle = \bigotimes_{i=1}^{L} |+\rangle_i \quad \text{where} \quad |+\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{C}^{M+2n} .$$

Then $\qquad t(\lambda) \mid \omega_+\rangle = \Lambda^0(\lambda)\ |\omega_+\rangle\ , \qquad$ where

$$\Lambda^0(\lambda) \quad = \quad a(\lambda)^{2L} g_0(\lambda) + b(\lambda)^{2L} \sum_{l=1}^{2n+M-2} (-1)^{[l+1]} g_l(\lambda) + c(\lambda)^{2L} g_{2n+M-1}(\lambda)$$

with

$$a(\lambda) = (\lambda+i)(\lambda+i\kappa), \qquad b(\lambda) = \lambda(\lambda+i\kappa), \qquad c(\lambda) = \lambda(\lambda+i\kappa-i)$$

the functions $g(\lambda)$ being written as (in the case $osp(2m+1|2n)$, with $K^{\pm} = \mathbb{I}$)

$$\begin{aligned}
g_l(\lambda) &= \frac{\lambda(\lambda+\frac{i\kappa}{2}-\frac{i}{2})(\lambda+i\kappa)}{(\lambda+\frac{i\kappa}{2})(\lambda+\frac{il}{2})(\lambda+\frac{i(l+1)}{2})}, \qquad l = 0,\ldots,n-1, \\
g_l(\lambda) &= \frac{\lambda(\lambda+\frac{i\kappa}{2}-\frac{i}{2})(\lambda+i\kappa)}{(\lambda+\frac{i\kappa}{2})(\lambda+in-\frac{il}{2})(\lambda+in-\frac{i(l+1)}{2})}, \\
&\qquad\qquad\qquad\qquad\qquad l = n,\ldots,n+m-1 \\
g_{n+m}(\lambda) &= \frac{\lambda(\lambda+i\kappa)}{(\lambda+i\frac{n-m}{2})(\lambda+i\frac{n-m+1}{2})} \qquad \text{if } M = 2m+1 \\
g_l(\lambda) &= g_{2n+M-l-1}(-\lambda-i\kappa), \qquad l = 0,1,...,M+2n
\end{aligned}$$

The dressing consists in the insertion of factors $A_l(\lambda)$ to get the other eigenvalues $\Lambda(\lambda)$ of the transfer matrix

$$\begin{aligned}
\Lambda(\lambda) &= a(\lambda)^{2L} g_0(\lambda) A_0(\lambda) + b(\lambda)^{2L} \sum_{l=1}^{2n+M-2} (-1)^{[l+1]} g_l(\lambda) A_l(\lambda) \\
&+ c(\lambda)^{2L} g_{2n+M-1}(\lambda) A_{2n+M-1}(\lambda)
\end{aligned}$$

The factors A_l take the form

$$
\begin{aligned}
A_0(\lambda) &= \prod_{j=1}^{M^{(1)}} \frac{\lambda+\lambda_j^{(1)}-\frac{i}{2}}{\lambda+\lambda_j^{(1)}+\frac{i}{2}} \, \frac{\lambda-\lambda_j^{(1)}-\frac{i}{2}}{\lambda-\lambda_j^{(1)}+\frac{i}{2}} \,, \\
A_l(\lambda) &= \prod_{j=1}^{M^{(l)}} \frac{\lambda+\lambda_j^{(l)}+\frac{il}{2}+i}{\lambda+\lambda_j^{(l)}+\frac{il}{2}} \, \frac{\lambda-\lambda_j^{(l)}+\frac{il}{2}+i}{\lambda-\lambda_j^{(l)}+\frac{il}{2}} \\
&\times \prod_{j=1}^{M^{(l+1)}} \frac{\lambda+\lambda_j^{(l+1)}+\frac{il}{2}-\frac{i}{2}}{\lambda+\lambda_j^{(l+1)}+\frac{il}{2}+\frac{i}{2}} \, \frac{\lambda-\lambda_j^{(l+1)}+\frac{il}{2}-\frac{i}{2}}{\lambda-\lambda_j^{(l+1)}+\frac{il}{2}+\frac{i}{2}} \\
& \quad l = 1, \ldots, n-1
\end{aligned}
$$

Analyticity around the poles introduced in the factors A_l now imposes the so-called Bethe equations in the λ_i :

$$
e_1(\lambda_i^{(1)})^{2L} = \prod_{\epsilon=\pm 1} \prod_{j=1, j\neq i}^{M^{(1)}} e_2(\lambda_i^{(1)} - \epsilon\lambda_j^{(1)}) \, \prod_{j=1}^{M^{(2)}} e_{-1}(\lambda_i^{(1)} - \epsilon\lambda_j^{(2)})
$$

$$
1 = \prod_{\epsilon=\pm 1} \prod_{j=1, j\neq i}^{M^{(l)}} e_2(\lambda_i^{(l)} - \epsilon\lambda_j^{(l)}) \, \prod_{\tau=\pm 1} \prod_{j=1}^{M^{(l+\tau)}} e_{-1}(\lambda_i^{(l)} - \epsilon\lambda_j^{(l+\tau)})
$$

$$
l = 2, \ldots, n+m-1, \;\; l \neq n
$$

$$
1 = \prod_{\epsilon=\pm 1} \prod_{j=1}^{M^{(n+1)}} e_1(\lambda_i^{(n)} - \epsilon\lambda_j^{(n+1)}) \, \prod_{j=1}^{M^{(n-1)}} e_{-1}(\lambda_i^{(n)} - \epsilon\lambda_j^{(n-1)})
$$

$$
1 = \prod_{\epsilon=\pm 1} \prod_{j=1, j\neq i}^{M^{(n+m)}} e_1(\lambda_i^{(n+m)} - \epsilon\lambda_j^{(n+m)}) \, \prod_{j=1}^{M^{(n+m-1)}} e_{-1}(\lambda_i^{(n+m)} - \epsilon\lambda_j^{(n+m-1)})
$$

with

$$
e_x(\lambda) = \frac{\lambda+\frac{ix}{2}}{\lambda-\frac{ix}{2}} \,.
$$

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